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Variations on the Chebyshev and L^q Theories of Best Approximation

A. PINKUS

Department of Mathematics, Technion, Haifa 32000, Israel

AND

O. SHISHA

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881, USA

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A new method of approximation is proposed which maintains many of the essentials of the classical theory of best uniform approximation, while also using an L^q-type measure of approximation.

1. INTRODUCTION

The classical Chebyshev theory of best uniform approximation to continuous functions by polynomials of degree $\leqslant n$ was initiated by Chebyshev in [2]. This theory has a distinct advantage over the corresponding ones for L^q-norms, $1 \leqslant q < \infty$, in that the unique best approximant is characterized by a remarkable geometric property. Let f be a real-valued function, continuous on $[0, 1]$, and, for $n = 0, 1, 2,\ldots$, let π_n denote the set of all real algebraic polynomials of degree at most n. Then $p^* \in \pi_n$ is the unique best uniform approximant to f from π_n if and only if there exist $n + 2$ points $0 \leqslant x_1 < \cdots < x_{n+2} \leqslant 1$, and a fixed $\sigma = \pm 1$, for which

$$\sigma(-1)^k\,(f-p^*)(x_k) = \max_{0 \leqslant x \leqslant 1} |f(x) - p^*(x)|, \qquad k = 1,\ldots, n+2.$$

On the other hand, for $q \in (1, \infty)$, $p^* \in \pi_n$ is the best $L^q[0, 1]$ approximant to f from π_n if and only if

$$\int_0^1 |(f-p^*)(x)|^{q-1}\,\mathrm{sgn}(f-p^*)(x)\,x^k\,dx = 0, \qquad k = 0, 1,\ldots, n,$$

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a condition which is difficult to check, and which lacks the simple geometric appeal of Chebyshev's characterization. For $q=1$, the situation is even slightly more complicated, but is essentially the same as for $q \in (1, \infty)$.

In this paper, we propose a new method of approximation which maintains the geometric flavour of Chebyshev's characterization of the best uniform approximation, while also using an L^q-type $(1 \leqslant q < \infty)$ measure of approximation. We have, however, to pay a price, and the foremost cost is that our "distance" function is *not* derived from a norm. However, this drawback is not all that costly. We, perhaps surprisingly, do maintain the uniqueness of a best approximant, and we are able to give a fairly simple characterization thereof.

Let $C[0,1]$ denote the class of real-valued functions continuous on $[0,1]$. For $f \in C[0,1]$, and $q \in [1, \infty)$, set

$$|||f|||_q = \sup \left(\int_a^b |f(x)|^q \, dx \right)^{1/q}, \tag{A}$$

where the supremum is taken over all a, b, $0 \leqslant a \leqslant b \leqslant 1$, for which $f(x) > 0$ on (a,b), or $f(x) < 0$ on (a,b). Let us also define

$$|||f|||_{q*} = \sup \left(\int_a^b |f(x)|^q \, dx \right)^{1/q}, \tag{B}$$

where the supremum is taken over all a, b, $0 \leqslant a \leqslant b \leqslant 1$, for which $f(x) \geqslant 0$ on (a,b), or $f(x) \leqslant 0$ thereon. Thus, if $f \equiv 0$, then $|||f|||_q = |||f|||_{q*} = 0$. The analogous definitions for $q = \infty$ and $f \in C[0,1]$ are simply $|||f|||_\infty = |||f|||_{\infty*} = \|f\|_\infty = \max\{|f(x)| : x \in [0,1]\}$.

With minor modifications, the results of this paper hold for all $q \in [1, \infty)$ (except for Theorem 4.1). However, for the sake of simplicity, we shall only deal with the case $q=1$, and for ease of notation we set, for every $f \in C[0,1]$,

$$|||f||| = |||f|||_1, \tag{1.1}$$

$$|||f|||_* = |||f|||_{1*}. \tag{1.2}$$

Before describing our results, let us note

LEMMA 1.1. *The suprema in* (A) *and* (B) *are attained.*

The proof is left to the reader.

We shall be concerned with the following two quantities:

$$\inf_{p \in \pi_n} |||f - p|||, \tag{1.3}$$

and

$$\inf_{p \in \pi_n} |||f-p|||_* . \tag{1.4}$$

We show (Theorem 2.5) that the infimum in (1.4) need not be attained, while that in (1.3) always is. Furthermore, a $p^* \in \pi_n$ which attains the minimum (infimum) in (1.3) is unique, and is characterized by the fact that there exist $n+2$ consecutive disjoint open subintervals $I_1, I_2, \ldots, I_{n+2}$ of $[0, 1]$, and $\sigma = \pm 1$, such that

$$(-1)^k \sigma (f-p^*)(x) \geqslant 0 \text{ throughout } I_k,$$

and

$$(-1)^k \sigma \int_{I_k} (f-p^*)(x)\, dx \geqslant |||f-p^*|||,$$

for $k = 1, \ldots, n+2$ (Theorem 3.1).

If the infimum in (1.4) is attained, then it is attained only by the p^* pertaining to (1.3). It is attained if and only if there exist $n+2$ consecutive disjoint open subintervals $I_1, I_2, \ldots, I_{n+2}$ of $[0, 1]$, and $\sigma = \pm 1$, such that

$$(-1)^k \sigma (f-p^*)(x) \geqslant 0 \text{ throughout } I_k,$$

and

$$(-1)^k \sigma \int_{I_k} (f-p^*)(x)\, dx = |||f-p^*|||_* ,$$

for $k = 1, 2, \ldots, n+2$ (Theorem 3.2).

Aside from considering the basic questions of existence, uniqueness and characterization, we also provide analogues of de La Vallée-Poussin's bound (Theorems 4.3 and 4.4) and of Bernstein's comparison theorem (Theorem 4.5), and determine the minimal "norm" monic polynomial of degree $n+1$ for $|||\cdot|||$ and $|||\cdot|||_*$ (Theorem 4.1).

All the approximation results which follow are stated for π_n. However, the analysis is such that these same results hold, *mutatis mutandis*, if we replace π_n by any Chebyshev system of order $n+1$.

2. Preliminaries and Existence Results

On the basis of Lemma 1.1, we can restate the definitions of $|||\cdot|||$ and $|||\cdot|||_*$:

DEFINITION 2.1. *For $f \in C[0, 1]$,*

$$|||f||| = \max \left\{ \left| \int_a^b f(x)\, dx \right| : 0 \leqslant a \leqslant b \leqslant 1, f(x) > 0 \text{ on } (a, b), \right.$$

$$\left. \text{or } f(x) < 0 \text{ on } (a, b) \right\}, \tag{2.1}$$

$$|||f|||_* = \max \left\{ \left| \int_a^b f(x)\, dx \right| : 0 \leqslant a \leqslant b \leqslant 1, f(x) \geqslant 0 \text{ on } (a, b), \right.$$

$$\left. \text{or } f(x) \leqslant 0 \text{ on } (a, b) \right\}. \tag{2.2}$$

Observe that the sets in (2.1) and (2.2) are nonempty, as a is allowed to equal b. From the definitions it follows that, for $f \in C[0, 1]$,

$$|||f||| \leqslant |||f|||_* \leqslant \|f\|_1 \leqslant \|f\|_\infty, \tag{2.3}$$

where $\|f\|_1 = \int_0^1 |f(x)|\, dx$, and, as before, $\|f\|_\infty = \max\{|f(x)| : x \in [0, 1]\}$. On the basis of Definition 2.1, the following result is evident.

LEMMA 2.2. *For every $f \in C[0, 1]$,*

(a) $|||f||| = 0$ *if and only if* $f(x) = 0$ *for all* $x \in [0, 1]$, *and similarly for* $|||f|||_*$.

(b) *If $f \in C[0, 1]$, and c is a real number, then*

$$|||cf||| = |c| \cdot |||f|||,$$
$$|||cf|||_* = |c| \cdot |||f|||_*.$$

Neither $|||\cdot|||$ nor $|||\cdot|||_*$ are norms.

PROPOSITION 2.3. *Neither $|||\cdot|||$ nor $|||\cdot|||_*$ satisfies the triangle inequality.*

Proof. Let $f(x) \equiv |x - (1/2)| - (1/8)$, and $g(x) \equiv (1/2) - |x - (1/2)|$. Then $|||f||| = |||f|||_* = 9/128$, $|||g||| = |||g|||_* = 1/4$, and $|||f+g||| = |||f+g|||_* = 3/8 > |||f||| + |||g||| = |||f|||_* + |||g|||_*$. On the other hand, it is very easy to construct particular $f, g \in C[0, 1]$ for which the triangle inequality does hold.

We were unable to find a standard terminology for functionals satisfying Lemma 2.2, but not necessarily satisfying the triangle inequality. Cwikel and Peetre [3] call similar functionals "gauges," and we also shall use this term.

Let f, $f_m \in C[0, 1]$, $m = 1, 2, \ldots$, and let f_m converge to f, uniformly on $[0, 1]$. From (2.3) it easily follows that

$$\lim_{m\to\infty} |||f_m - f||| = \lim_{m\to\infty} |||f_m - f|||_* = 0. \tag{2.4}$$

However, it does not follow that $\lim_{m\to\infty} |||f_m||| = |||f|||$, or $\lim_{m\to\infty} |||f_m|||_* = |||f|||_*$. An important property of $|||\cdot|||$ which will be used to prove the existence of a $|||\cdot|||$-best polynomial approximation is concerned with such limit questions.

THEOREM 2.4. *Assume that f, $f_m \in C[0, 1]$, $m = 1, 2, \ldots$, and f_m tends to f, uniformly on $[0, 1]$. Then*

$$|||f||| \leqslant \varliminf_{m\to\infty} |||f_m||| \leqslant \varlimsup_{m\to\infty} |||f_m|||_* \leqslant |||f|||_*. \tag{2.5}$$

The middle inequality is an immediate consequence of (2.3). To show that equality need not hold in the other two inequalities, consider the following two examples.

Let $f(x) \equiv |x - (1/2)|$. Thus $|||f||| = 1/8$, and $|||f|||_* = 1/4$. Set $f_m(x) \equiv |x - (1/2)| - (1/m)$. Then $|||f_m|||_* < 1/8$ for all $m \geqslant 3$, which implies that

$$\varlimsup_{m\to\infty} |||f_m|||_* = 1/8 < |||f|||_*.$$

On the other hand, set $\tilde{f}_m(x) \equiv |x - (1/2)| + (1/m)$, $m = 1, 2, \ldots$. Then $|||\tilde{f}_m||| > 1/4$ for all $m \geqslant 1$, so that

$$\varliminf_{m\to\infty} |||\tilde{f}_m||| = 1/4 > |||f|||.$$

Proof of Theorem 2.4. We shall first prove the more important inequality, namely,

$$|||f||| \leqslant \varliminf_{m\to\infty} |||f_m|||.$$

If $f \equiv 0$, then equality holds, by (2.4). Thus we assume $f \not\equiv 0$. Let $\sigma = \pm 1$, and $0 \leqslant a < b \leqslant 1$ be such that

$$|||f||| = \sigma \int_a^b f(x)\, dx,$$

where $\sigma f(x) > 0$ for all $x \in (a, b)$. Let $\varepsilon > 0$, and choose $\delta > 0$ ($\delta < (b - a)/2$) so that

$$\sigma \int_{a+\delta}^{b-\delta} f(x)\, dx > |||f||| - (\varepsilon/2).$$

Since σf is bounded away from zero on $[a+\delta, b-\delta]$, and f_m tends uniformly to f on $[0, 1]$, there exists an M such that, for all $m \geqslant M$, $\sigma f_m(x) > 0$ on $(a+\delta, b-\delta)$, and

$$\sigma \int_{a+\delta}^{b-\delta} f_m(x)\, dx \geqslant |||f||| - \varepsilon.$$

Thus $|||f_m||| \geqslant |||f||| - \varepsilon$, and therefore

$$|||f||| \leqslant \varliminf_{m\to\infty} |||f_m|||.$$

To prove the remaining inequality, we again assume that $f \not\equiv 0$. Suppose, to the contrary, that

$$\varlimsup_{m\to\infty} |||f_m|||_* > |||f|||_*.$$

Thus f_m contains a subsequence, which we again denote by f_m, for which

$$\lim_{m\to\infty} |||f_m|||_* = |||f|||_* + c,$$

$c > 0$ being a constant. Hence, for every $m \geqslant$ some M', there exists an interval $[a_m, b_m]$, and a $\sigma_m = \pm 1$, such that $\sigma_m f_m(x) \geqslant 0$ throughout $[a_m, b_m]$, and

$$\sigma_m \int_{a_m}^{b_m} f_m(x)\, dx \geqslant |||f|||_* + (c/2).$$

By passing to a subsequence, if necessary, we may assume that all $\sigma_m = \sigma = +1$ or -1, fixed. Take a convergent subsequence of a_m, and one of b_m (again denoted a_m, b_m), say, $a_m \to a$, $b_m \to b$. Thus, $\sigma f(x) \geqslant 0$ throughout $[a, b]$, and

$$\sigma \int_a^b f(x)\, dx \geqslant |||f|||_* + (c/2).$$

But

$$\sigma \int_a^b f(x)\, dx \leqslant |||f|||_*,$$

a contradiction.

We are concerned with the problem of approximating f from π_n, using the gauges $|||\cdot|||$ and $|||\cdot|||_*$. The next result is an almost immediate consequence of (2.5). However, for completeness, we include the proof.

THEOREM 2.5. *Let $f \in C[0,1]$, and $n \geqslant 0$.*

(a) *There exists a $p^* \in \pi_n$ for which*

$$\inf_{p \in \pi_n} |||f-p||| = |||f-p^*|||.$$

(b) $\inf_{p \in \pi_n} |||f-p|||_*$ *may be unattained.*

Proof. We first prove (b). Let $f \in C[0,1]$ be defined to be linear in each of the intervals $(i/4, (i+1)/4)$, $i = 0, 1, 2, 3$, and to satisfy $f(0) = 4$, $f(1/4) = f(3/4) = 0$, $f(1/2) = 2$, and $f(1) = -6$. Its graph is shown in Fig. 1.

Thus $\int_0^{1/4} f(x)\,dx = 1/2$, $\int_{1/4}^{3/4} f(x)\,dx = 1/2$, and $\int_{3/4}^{1} f(x)\,dx = -3/4$, so that $|||f|||_* = 1$. Consider $|||f-c|||_*$ for a given real c. If $c \leqslant 0$, then $|||f-c|||_* \geqslant 1$. If $c > 0$, then $|||f-c|||_* > 3/4$, and, clearly, $|||f-(1/n)|||_* \to 3/4$. So

$$\inf_{p \in \pi_0} |||f-p|||_* = \inf_{c \in \mathbb{R}} |||f-c|||_* = 3/4$$

which is not attained.

Part (a) is a consequence of the lower semicontinuity of $|||\cdot|||$. Set

$$\inf_{p \in \pi_n} |||f-p||| = C.$$

We may assume that $0 < C < |||f|||$. For $m = 1, 2, \ldots$, let $p_m(x) \equiv \sum_{k=0}^{n} a_k^{(m)} x^k \not\equiv 0$ satisfy

$$\lim_{m \to \infty} |||f-p_m||| = C,$$

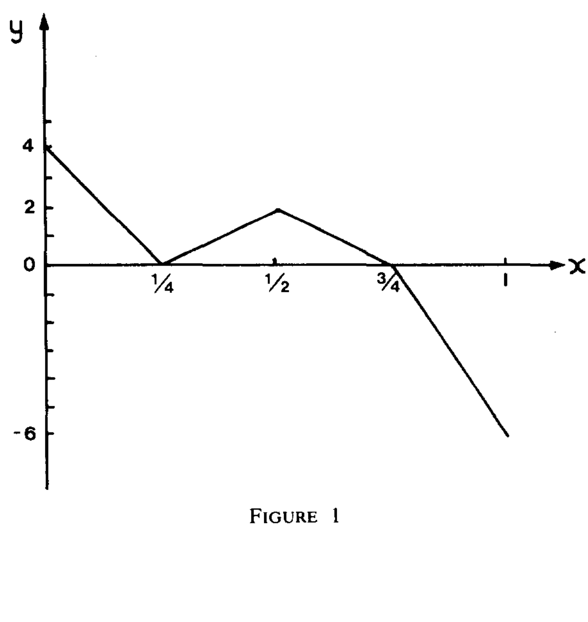

FIGURE 1

and let $\mu_m = \max\{|a_k^{(m)}|: 0 \leqslant k \leqslant n\} > 0$. We first prove that μ_m is a bounded sequence. If this is not the case, then there exists a subsequence, again denoted by μ_m, which tends to ∞. By choosing suitable subsequences, we may assume that, for $m = 1, 2, \ldots$, $\mu_m = |a_{k_0}^{(m)}|$, with a fixed k_0, and that, for $k = 0, 1, \ldots, n$, $a_k^{(m)}/\mu_m$ converges, say, to a_k, $|a_k| \leqslant 1$. Set $p(x) \equiv \sum_{k=0}^n a_k x^k$, and

$$g_m(x) \equiv \mu_m^{-1}(f(x) - p_m(x)), \qquad m = 1, 2, \ldots.$$

Then $g_m(x)$ tends uniformly to $-p(x)$ on $[0, 1]$. Since $p(x) \not\equiv 0$, $|||p||| > 0$. By (2.5),

$$\varliminf_{m \to \infty} |||g_m||| \geqslant |||p||| > 0.$$

However,

$$\lim_{m \to \infty} |||g_m||| = \lim_{m \to \infty} \mu_m^{-1} |||f - p_m||| = 0.$$

This contradiction proves that μ_m is bounded.

Hence there are integers $1 \leqslant m_1 < m_2 < \ldots$, and reals $a_0, a_1, \ldots, a_n$ for which $\lim_{j \to \infty} a_k^{(m_j)} = a_k$, $k = 0, 1, \ldots, n$. Thus

$$\lim_{j \to \infty} p_{m_j}(x) = p^*(x) = \sum_{k=0}^n a_k x^k,$$

uniformly in $[0, 1]$. By (2.5),

$$C = \lim_{j \to \infty} |||f - p_{m_j}||| \geqslant |||f - p^*|||.$$

The definition of C implies that

$$\inf_{p \in \pi_n} |||f - p||| = |||f - p^*|||.$$

3. Characterization and Uniqueness

If I and J are subintervals of $[0, 1]$, then by $I < J$ we mean that $x < y$ for all $x \in I$, and $y \in J$. We shall also use the notation $I < x < J$, with the obvious meaning. With this understanding, we now state our first main result.

THEOREM 3.1. *Let $f \in C[0, 1]$, and $n \geqslant 0$. Then there is a unique $p^* \in \pi_n$ satisfying*

$$\inf_{p \in \pi_n} |||f - p||| = |||f - p^*|||. \tag{3.1}$$

This p^* is characterized by the following property:
There exist $n+2$ disjoint (nonempty) open intervals

$$I_1 < \cdots < I_{n+2},$$

and $\sigma = +1$ or -1, fixed, such that, for $k = 1,..., n+2$,

(a) $(-1)^k \sigma(f-p^*) \geqslant 0$ *on* I_k,
(b) $(-1)^k \sigma \int_{I_k} (f-p^*)(x)\, dx \geqslant |||f-p^*|||$.

Proof. Assume that $p^* \in \pi_n$ satisfies (a) and (b). Let $p \in \pi_n$, $p \not\equiv p^*$. We shall prove that $|||f-p||| > |||f-p^*|||$. Suppose $|||f-p||| \leqslant |||f-p^*|||$. We claim that there exist $x_k \in I_k$, $k = 1,..., n+2$, for which

$$(-1)^k \sigma(p-p^*)(x_k) \geqslant 0, \qquad k = 1,..., n+2. \tag{3.2}$$

But from (3.2) it is not difficult to infer that $p - p^* \not\equiv 0$ has at least $n+1$ zeros, counting multiplicities, which is false. If, for some $k \in \{1,..., n+2\}$, $(-1)^k \sigma(p-p^*)(x) < 0$, for all $x \in I_k$, then, by (a),

$$(-1)^k \sigma(f-p)(x) > (-1)^k \sigma(f-p^*)(x) \geqslant 0$$

for all $x \in I_k$, and hence,

$$(-1)^k \sigma \int_{I_k} (f-p^*)(x)\, dx < (-1)^k \sigma \int_{I_k} (f-p)(x)\, dx \leqslant |||f-p||| \leqslant |||f-p^*|||,$$

contradicting (b).

Assume now that $p^* \in \pi_n$ satisfies (3.1). We shall prove the existence of $I_1,..., I_{n+2}$ as in the theorem. We may assume $f - p^* \not\equiv 0$.

A *maximal-definite* interval is an $I = (\alpha, \beta)$, $0 \leqslant \alpha < \beta \leqslant 1$, which, for some $\sigma = \pm 1$ (the signum of I), satisfies:

(1) $\sigma(f-p^*) \geqslant 0$ on I;

(2) $\sigma \int_I (f-p^*)(x)\, dx \geqslant |||f-p^*|||$;

(3) if J is an open subinterval of (0, 1), $I \subset J$, and $\sigma(f-p^*) \geqslant 0$ on J, then $f - p^* = 0$ on $J \backslash I$;

(4) there is no open, nonempty subinterval of I having α or β as an endpoint, throughout which $f - p^* = 0$.

By straightforward arguments, there exists a maximal-definite interval, the set of such intervals is finite, and they are all mutually disjoint. Let them be

$$J_1 < \cdots < J_m.$$

Let

$$J_1, \ldots, J_{m_1}$$

have the same signum σ_1,

$$J_{m_1+1}, \ldots, J_{m_2}$$

have the same signum $-\sigma_1$, and so on, until we reach

$$J_{m_{s-1}+1}, \ldots, J_{m_s} \qquad (m_s = m),$$

having the same signum. We wish to prove that $s \geqslant n+2$. Suppose not. We may assume

$$(-1)^{k+1} (f - p^*)(x) \geqslant 0 \text{ throughout } J_l,$$

whenever $m_{k-1} + 1 \leqslant l \leqslant m_k$, $k = 1, \ldots, s$, where $m_0 = 0$ (for otherwise, we consider $-f$). For $j = 1, \ldots, s-1$ (in case $s > 1$), choose an x_j satisfying $J_{m_j} < x_j < J_{m_j+1}$, and $(f - p^*)(x_j) = 0$. Such an x_j exists since

$$(-1)^{j+1} (f - p^*) \geqslant 0 \qquad \text{on} \quad J_{m_j},$$

and

$$(-1)^{j+1} (f - p^*) \leqslant 0 \qquad \text{on} \quad J_{m_j+1}.$$

Set

$$p(x) \equiv \prod_{j=1}^{s-1} (x_j - x) \qquad (\equiv 1 \text{ if } s = 1),$$

and let $x_0 = 0$, $x_s = 1$. Since $s < n+2$, $p \in \pi_n$.

We shall prove that there exists an $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$,

$$|||f - p^* - \varepsilon p||| < |||f - p^*|||,$$

contradicting the definition of p^*.

Since $(f - p^* - \varepsilon p)(x_j) = 0$ whenever $1 \leqslant j \leqslant s-1$, for all real ε, it follows from (2.1) that

$$|||f - p^* - \varepsilon p||| = \max_{1 \leqslant j \leqslant s} |||f - p^* - \varepsilon p|||_{[x_{j-1}, x_j]},$$

where, on the right-hand side, the underlying interval for the gauge is

$[x_{j-1}, x_j]$ rather than $[0, 1]$. It therefore suffices to prove that, for $j = 1,\ldots, s$, there exists an $\varepsilon^{(j)} > 0$ such that, for all $\varepsilon \in (0, \varepsilon^{(j)})$,

$$|||f - p^* - \varepsilon p|||_{[x_{j-1}, x_j]} < |||f - p^*|||.$$

Let $j \in \{1,\ldots, s\}$, and suppose no such $\varepsilon^{(j)}$ exists. Then there exist $\varepsilon_1, \varepsilon_2,\ldots,$ all > 0, $\varepsilon_n \to 0$, such that, for $n = 1, 2,\ldots,$

$$|||f - p^* - \varepsilon_n p|||_{[x_{j-1}, x_j]} \geqslant |||f - p^*|||.$$

Therefore there exists, for $n = 1, 2,\ldots,$ an open subinterval I_n of $[x_{j-1}, x_j]$, and $\sigma_n = \pm 1$, for which

$$\sigma_n(f - p^* - \varepsilon_n p) \geqslant 0 \qquad \text{on} \quad I_n,$$

and

$$\sigma_n \int_{I_n} (f - p^* - \varepsilon_n p)(x)\, dx \geqslant |||f - p^*|||.$$

On (x_{j-1}, x_j), $(-1)^{j+1} p(x) > 0$, and throughout each $J_k \subseteq [x_{j-1}, x_j]$, $(-1)^{j+1}(f - p^*)(x) \geqslant 0$. If $\sigma_n = (-1)^{j+1}$ for some n, then, on I_n, $0 \leqslant (-1)^{j+1}(f - p^* - \varepsilon_n p) < (-1)^{j+1}(f - p^*)$, and, hence,

$$\begin{aligned} |||f - p^*||| &\leqslant (-1)^{j+1} \int_{I_n} (f - p^* - \varepsilon_n p)(x)\, dx \\ &< (-1)^{j+1} \int_{I_n} (f - p^*)(x)\, dx \\ &\leqslant |||f - p^*|||. \end{aligned}$$

Thus $\sigma_n = (-1)^j$ for all n.

For $n = 1, 2,\ldots,$ let $I_n = (a_n, b_n)$, so that $x_{j-1} \leqslant a_n < b_n \leqslant x_j$. There exist integers $1 \leqslant n_1 < n_2 < \cdots$ such that $a_{n_k} \to a$, and $b_{n_k} \to b$. It follows that

$$(-1)^j (f - p^*) \geqslant 0 \qquad \text{on} \quad (a, b),$$

and

$$(-1)^j \int_a^b (f - p^*)(x)\, dx \geqslant |||f - p^*|||.$$

Hence $(a, b) \subseteq (x_{j-1}, x_j)$ must intersect some maximal-definite subinterval J_k of $[x_{j-1}, x_j]$ at a point where $f - p^* \neq 0$. A contradiction now ensues, since, at such a point, the above discussion implies that $\operatorname{sgn}(f - p^*)$ is both $(-1)^j$ and $(-1)^{j+1}$. This completes the proof of the theorem.

We shall now consider an analogue of Theorem 3.1 for the gauge $|||\cdot|||_*$. We prove that, for a fixed n, $\inf\{|||f-p|||_* : p \in \pi_n\}$, if attained, is uniquely attained by the p^* of Theorem 3.1.

THEOREM 3.2. *Let $f \in C[0,1]$, and $n \geqslant 0$. Then*

$$\inf_{p \in \pi_n} |||f-p|||_* \tag{3.3}$$

is attained if and only if there exist $n+2$ disjoint (nonempty) open intervals $I_1 < \cdots < I_{n+2}$, and $\sigma = +1$ or -1, fixed, such that, for $k = 1,..., n+2$,

(a) $(-1)^k \sigma(f-p^*) \geqslant 0$ *on* I_k,

(b) $(-1)^k \sigma \int_{I_k} (f-p^*)(x)\, dx = |||f-p^*|||_*$,

where p^* is the unique minimal polynomial given in Theorem 3.1. *If the infimum in* (3.3) *is attained, then it is attained by p^* only.*

Proof. It follows from Theorem 3.1 that only p^* of that theorem can satisfy conditions (a) and (b) of Theorem 3.2. Namely, if $\tilde{p} \in \pi_n$ satisfies, for $k = 1,..., n+2$,

(a′) $(-1)^k \sigma(f-\tilde{p}) \geqslant 0$ on I_k,

(b′) $(-1)^k \sigma \int_{I_k} (f-\tilde{p})(x)\, dx = |||f-\tilde{p}|||_*$,

where $I_1 < \cdots < I_{n+2}$ are disjoint (nonempty) open intervals of $[0,1]$, and $\sigma = \pm 1$, then, since $|||f-\tilde{p}|||_* \geqslant |||f-\tilde{p}|||$, it follows by Theorem 3.1 that $\tilde{p} = p^*$. Assume that p^* does satisfy these conditions. The fact that p^* is the unique minimum approximant with respect to the gauge $|||\cdot|||_*$ is proven in exactly the same way as was the analogous result of Theorem 3.1. It thus remains to prove that if the infimum (3.3) is attained by some $\tilde{p} \in \pi_n$, then $\tilde{p}$ satisfies conditions (a′) and (b′).

We may assume $f - \tilde{p} \not\equiv 0$. Analogously to the definition given in the proof of Theorem 3.1, we define a *definite* interval to be an $I = (\alpha, \beta)$, $0 \leqslant \alpha < \beta \leqslant 1$, which, for some $\sigma = \pm 1$ (the signum of I), satisfies

(1) $\sigma(f-\tilde{p}) \geqslant 0$ on I;

(2) $\sigma \int_I (f-\tilde{p})(x)\, dx = |||f-\tilde{p}|||_*$;

(3) no open proper subinterval of I satisfies (1) and (2).

Again, the set of definite intervals is finite but nonempty, and they are mutually disjoint. Let them be

$$J_1 < \cdots < J_m.$$

As in the proof of Theorem 3.1, assume that

$$J_1, \ldots, J_{m_1}$$

have the same signum σ_1,

$$J_{m_1+1}, \ldots, J_{m_2}$$

have the same signum $-\sigma_1$, and so on, until we reach

$$J_{m_{s-1}+1}, \ldots, J_{m_s} \qquad (m_s = m),$$

having the same signum. We shall prove that $s \geqslant n+2$.

Suppose $s < n+2$, and assume that

$$(-1)^{k+1}(f-\tilde{p})(x) \geqslant 0 \qquad \text{on} \quad J_l,$$

whenever $m_{k-1}+1 \leqslant l \leqslant m_k$, $k = 1,\ldots, s$, where $m_0 = 0$ (otherwise consider $-f$). We proceed as in the proof of Theorem 3.1. As therein, there exists, for $j = 1,\ldots, s-1$ (in case $s > 1$), a point x_j satisfying $J_{m_j} < x_j < J_{m_j+1}$, and $(f-\tilde{p})(x_j) = 0$. Set, again,

$$p(x) \equiv \prod_{j=1}^{s-1} (x_j - x) \qquad (\equiv 1 \text{ if } s = 1).$$

Thus $p \in \pi_n$. Entirely analogously to the reasoning of the proof of Theorem 3.1, we can show that, for $j = 1,\ldots, s$, there exists an $\varepsilon^{(j)} > 0$ such that, for all $\varepsilon \in (0, \varepsilon^{(j)})$, $|||f-\tilde{p}-\varepsilon p|||_{*,[x_{j-1},x_j]} < |||f-\tilde{p}|||_*$, where $x_0 = 0$, and $x_s = 1$, with an obvious meaning for the left-hand side of the inequality.

However, our problem is that it is *not*, in general, true that

$$\max_{1 \leqslant j \leqslant s} |||f-\tilde{p}-\varepsilon p|||_{*,[x_{j-1},x_j]} = |||f-\tilde{p}-\varepsilon p|||_*.$$

To explain, if y is some fixed point in $(0, 1)$, and if $g \in C[0, 1]$ takes on both positive and negative values in every neighborhood of y, then we do have

$$|||g|||_* = \max\{|||g|||_{*,[0,y]}, |||g|||_{*,[y,1]}\}. \tag{3.4}$$

However, (3.4) may fail to hold if we merely know that $g(y) = 0$. It is this problem which we must deal with in considering $|||f-\tilde{p}-\varepsilon p|||_*$.

Let us first dispose of those cases wherein $|||\cdot|||_*$ can be "decomposed" as in (3.4). Suppose $1 \leqslant j \leqslant s-1$. Our only requirements on the point x_j were

$J_{m_j} < x_j < J_{m_j+1}$, and $(f-\tilde{p})(x_j)=0$. Assume now that $\bar{J}_{m_j} < \bar{J}_{m_j+1}$. We claim that we can choose x_j so that

$$|||f-\tilde{p}-\varepsilon p|||_* = \max\{|||f-\tilde{p}-\varepsilon p|||_{*,[0,x_j]}, |||f-\tilde{p}-\varepsilon p|||_{*,[x_j,1]}\},$$
$$\text{for all } \varepsilon > 0.$$

To prove this, set $J_{m_j}=(\alpha_j, \beta_j)$, $J_{m_j+1}=(\gamma_j, \delta_j)$. Since $\bar{J}_{m_j} < \bar{J}_{m_j+1}$, $\beta_j < \gamma_j$. If there exist a, b, $\beta_j \leqslant a < b \leqslant \gamma_j$, for which $f-\tilde{p} \equiv 0$ on $[a, b]$, then choose x_j to be any point of (a, b), and our claim is easily verified. Assume now that this is not the case. By assumption,

$$(-1)^{j+1}(f-\tilde{p}) \geqslant 0 \qquad \text{on} \quad J_{m_j},$$
$$(-1)^{j}(f-\tilde{p}) \geqslant 0 \qquad \text{on} \quad J_{m_j+1},$$

and J_{m_j}, J_{m_j+1} are definite intervals. Thus there exist points z_1, z_2 for which $\beta_j < z_1 < z_2 < \gamma_j$, and

$$(-1)^{j}(f-\tilde{p})(z_1) > 0,$$
$$(-1)^{j+1}(f-\tilde{p})(z_2) > 0.$$

Set

$$x_j = \sup\{x: x < z_2, (-1)^j (f-\tilde{p})(x) > 0\}.$$

Clearly

(i) $(f-\tilde{p})(x_j)=0$;

(ii) for every $\delta > 0$, there exists a $y_1 \in (x_j-\delta, x_j)$, and a $y_2 \in (x_j, x_j+\delta)$ for which

$$(-1)^{j}(f-\tilde{p})(y_1) > 0,$$
$$(-1)^{j+1}(f-\tilde{p})(y_2) > 0.$$

Then

$$(-1)^{j+1}p(x) > 0 \qquad \text{on} \quad (x_j-\delta, x_j),$$

and

$$(-1)^{j}p(x) > 0 \qquad \text{on} \quad (x_j, x_j+\delta).$$

Thus, for all $\varepsilon > 0$, $f-\tilde{p}-\varepsilon p$ assumes both positive and negative values in each neighborhood of x_j.

We have therefore reduced our problem to the following situation: $J_{m_j}=$

(α_j, β_j), $J_{m_j+1} = (\beta_j, \delta_j)$ (and thus $x_j = \beta_j$), and there exist positive $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \to 0$, such that, for $n = 1, 2, \ldots$, there is an open interval $I_n \ni x_j$, and a $\sigma_n = \pm 1$ satisfying

(i) $\sigma_n(f - \tilde{p} - \varepsilon_n p) \geqslant 0$ on I_n,

(ii) $\sigma_n \int_{I_n} (f - \tilde{p} - \varepsilon_n p)(x)\, dx \geqslant |||f - \tilde{p}|||_*$.

If we can prove that such a situation cannot occur, then we will have proven our theorem.

For $n = 1, 2, \ldots$, set $I_n = (a_n, b_n)$, and assume, without loss of generality, that $\sigma_n = (-1)^{j+1}$. We shall first prove that $b_n \to \beta_j$ $(=x_j)$.

For every $\delta > 0$, there exists a $z_\delta \in (\beta_j, \beta_j + \delta)$ for which $(-1)^j (f - \tilde{p})(z_\delta) > 0$, since J_{m_j} and J_{m_j+1} are definite intervals. Thus, for n sufficiently large, $(-1)^j\ (f - \tilde{p} - \varepsilon_n p)(z_\delta) > 0$, and, hence, $\beta_j < b_n < z_\delta < \beta_j + \delta$. Thus $b_n \to \beta_j$. In a similar manner, it can be proven that, if $a_n < \alpha_j$ for an infinite number of n (which is only possible if $x_{j-1} = \alpha_j$), then $a_n \to \alpha_j$.

Before evaluating $(-1)^{j+1} \int_{I_n} (f - \tilde{p} - \varepsilon_n p)(x)\, dx$, let us estimate from above $(-1)^{j+1} \int_{\beta_j}^{b_n} (f - \tilde{p} - \varepsilon_n p)(x)\, dx$. We have $(-1)^{j+1} (f - \tilde{p} - \varepsilon_n p) \geqslant 0$ on I_n, and $(-1)^j p > 0$ and $(-1)^j (f - \tilde{p}) \geqslant 0$ on $(\beta_j, b_n) \subseteq J_{m_j+1}$. Therefore, on (β_j, b_n), $|f - \tilde{p} - \varepsilon_n p| \leqslant \varepsilon_n |p|$, and thus

$$\begin{aligned} (-1)^{j+1} \int_{\beta_j}^{b_n} (f - \tilde{p} - \varepsilon_n p)(x)\, dx &\leqslant \varepsilon_n \int_{\beta_j}^{b_n} |p(x)|\, dx \\ &\leqslant \varepsilon_n \|p\|_\infty (b_n - \beta_j) \\ &= o(\varepsilon_n). \end{aligned}$$

Similarly, if $a_n < \alpha_j$ for an infinite number of n, then

$$(-1)^{j+1} \int_{a_n}^{\alpha_j} (f - \tilde{p} - \varepsilon_n p)(x)\, dx = o(\varepsilon_n),$$

and, hence,

$$\begin{aligned} &(-1)^{j+1} \int_{I_n} (f - \tilde{p} - \varepsilon_n p)(x)\, dx \\ &\qquad = (-1)^{j+1} \int_{I_n \cap [\alpha_j, \beta_j]} (f - \tilde{p} - \varepsilon_n p)(x)\, dx + o(\varepsilon_n). \end{aligned}$$

It is easily seen that $a^* = \overline{\lim}_{n \to \infty} a_n \in [\alpha_j, \beta_j)$. Since $(-1)^{j+1} (f - \tilde{p} - \varepsilon_n p)$, $(-1)^{j+1} (f - \tilde{p})$ and $(-1)^{j+1} p$ are all nonnegative on $[\alpha_j, \beta_j]$, we obtain

$$\begin{aligned} (-1)^{j+1} \int_{I_n \cap [\alpha_j, \beta_j]} (f - \tilde{p} - \varepsilon_n p)(x)\, dx &\leqslant (-1)^{j+1} \int_{\alpha_j}^{\beta_j} (f - \tilde{p})(x)\, dx - \varepsilon_n C \\ &= |||f - \tilde{p}|||_* - \varepsilon_n C, \end{aligned}$$

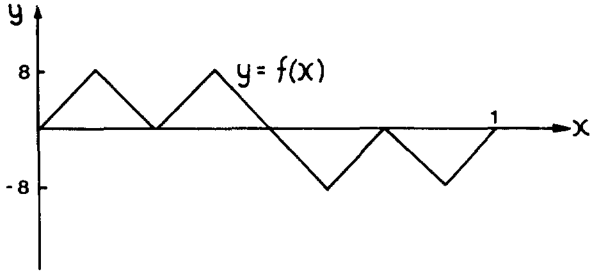

FIGURE 2

where $C = \int_{a^*}^{\beta_j} |p(x)|\,dx$, a fixed positive constant. Thus

$$(-1)^{j+1} \int_{I_n} (f - \tilde{p} - \varepsilon_n p)(x)\,dx \leqslant |||f - \tilde{p}|||_* - \varepsilon_n C + o(\varepsilon_n),$$

and therefore, for n sufficiently large,

$$(-1)^{j+1} \int_{I_n} (f - \tilde{p} - \varepsilon_n p)(x)\,dx < |||f - \tilde{p}|||_*.$$

Theorem 3.2 is now completely established.

Remark. Theorem 3.2 states that the infimum (3.3) is attained if and only if it is attained by the p^* of Theorem 3.1. This does not imply that if (3.3) is attained, then $|||f - p^*||| = |||f - p^*|||_*$. To see this, consider f of Fig. 2, where each of the four isosceles triangles has base 1/4, and area 1. Here $\inf\{|||f - p|||\colon\ p \in \pi_0\} = |||f||| = 1$, while $\inf\{|||f - p|||_* : p \in \pi_0\} = |||f|||_* = 2$.

4. ADDITIONAL RESULTS

We have thus far proven (or disproven) existence, uniqueness and characterization results for best polynomial approximation with respect to $|||\cdot|||$ and $|||\cdot|||_*$, analogous to well-known theorems for $\|\cdot\|_\infty$. In this section we prove analogues, for $|||\cdot|||$ and $|||\cdot|||_*$, of other known results concerning best approximation in L^∞.

Our first theorem identifies the monic polynomial of degree $n+1$ with minimal gauge. For $n = 0, 1, 2, \ldots$, let $\tilde{T}_n$ and $\tilde{U}_n$ denote the monic Chebyshev polynomials of the first and second kind, respectively, of degree n, on $[0, 1]$.

THEOREM 4.1. *Let $n \geqslant 0$. Then*

$$\min_{p \in \pi_n} |||x^{n+1} - p||| = \min_{p \in \pi_n} |||x^{n+1} - p|||_* = |||\tilde{U}_{n+1}|||$$
$$= \frac{2}{n+2} \|\tilde{T}'_{n+2}\|_\infty = \frac{2^{-2n-2}}{n+2}.$$

Proof. It is known that $\|\tilde{T}_{n+2}\|_\infty = 2^{-2n-3}$. The polynomial $\tilde{T}_{n+2}(x)$ is monic, has $n+3$ points of equioscillation $0 = x_0 < x_1 < \cdots < x_{n+2} = 1$, and is strictly monotone on (x_{j-1}, x_j), $j = 1,..., n+2$. Therefore $\tilde{U}_{n+1}(x) \equiv \tilde{T}'_{n+2}(x)/(n+2)$ is monic, and has the signum $(-1)^{j+n}$ on (x_{j-1}, x_j), for $j = 1,..., n+2$. Thus

$$(-1)^{j+n} \int_{x_{j-1}}^{x_j} \tilde{U}_{n+1}(x)\, dx = \frac{(-1)^{j+n}}{n+2} \int_{x_{j-1}}^{x_j} \tilde{T}'_{n+2}(x)\, dx$$
$$= \frac{(-1)^{j+n}}{n+2} [\tilde{T}_{n+2}(x_j) - \tilde{T}_{n+2}(x_{j-1})]$$
$$= \frac{2}{n+2} \|\tilde{T}_{n+2}\|_\infty,$$

for $j = 1,..., n+2$. The result now follows from Theorems 3.1 and 3.2.

Remarks. (a) $\tilde{U}_{n+1}(x)$ is also the unique monic polynomial of degree $n+1$ of minimum norm in $L^1[0, 1]$. However, it is not this which implies Theorem 4.1. The latter follows, as its proof shows, from the fact that $\tilde{U}_{n+1}(x)$ is the derivative of the $L^\infty[0, 1]$ minimal monic polynomial of degree $n+2$.

(b) From Theorem 4.1 we easily obtain the following extremal property of $\tilde{T}_n(x)$. Let $n \geqslant 1$, and set, for every $p \in \pi_n$, $\langle\langle p \rangle\rangle = \max\{|p(b) - p(a)|: 0 \leqslant a \leqslant b \leqslant 1, p \text{ is monotone on } [a, b]\}$. Then $\min\{\langle\langle p \rangle\rangle: p \in \pi_n, p \text{ monic}\} = \langle\langle \tilde{T}_n \rangle\rangle$, and $\tilde{T}_n(x)$ uniquely attains this minimum.

Let $n \geqslant 0$ be fixed. For each $f \in C[0, 1]$, let $p(f; x) \in \pi_n$ denote the unique polynomial which attains the minimum in (3.1). Since the gauge $|||\cdot|||$ is not a continuous mapping of $C[0, 1]$ into the reals (see the sentence following (2.4)), it is natural to ask whether $p(f; x)$ can be viewed as a continuous mapping of $C[0, 1]$ into itself.

THEOREM 4.2. *Let f, $f_m \in C[0, 1]$, $m = 1, 2,...,$ and assume that f_m converges uniformly to f on $[0, 1]$. Then $p(f_m; x)$ converges uniformly to $p(f; x)$ on $[0, 1]$.*

Proof. Since $\|f_m\|_\infty$ is bounded, so is $\|p(f_m;\cdot)\|_\infty$. Set

$$p(f_m;x) \equiv \sum_{k=0}^{n} a_k^{(m)} x^k, \qquad m = 1, 2, \ldots.$$

Then $a_k^{(m)}$ is bounded, for $k = 0, 1, \ldots, n$. Since $p(f_m;x)$ is uniformly bounded and (as is easily seen) equicontinuous on $[0, 1]$, it has a subsequence $p(f_{m_l};x)$ which converges uniformly to some $\tilde{p}(x) \in \pi_n$ on $[0, 1]$. We shall prove that $\tilde{p}(x) \equiv p(f;x)$. Since this will be true for every uniformly convergent subsequence, the result follows.

We may assume that $p(f_m;x)$ itself converges uniformly on $[0, 1]$ to $\tilde{p}(x) \not\equiv f(x)$. By Theorem 3.1, for $m = 1, 2, \ldots$, there exist disjoint (nonempty) open intervals of $[0, 1]$: $I_1^{(m)} < \cdots < I_{n+2}^{(m)}$, and a $\sigma_m = \pm 1$, for which

$$(-1)^k \sigma_m (f_m(x) - p(f_m;x)) \geqslant 0 \qquad \text{on} \quad I_k^{(m)},$$

and

$$(-1)^k \sigma_m \int_{I_k^{(m)}} (f_m(x) - p(f_m;x))\, dx \geqslant |||f_m - p(f_m;\cdot)|||,$$

for $k = 1, \ldots, n+2$. Let $I_k^{(m)} = (\alpha_k^{(m)}, \beta_k^{(m)})$, $k = 1, \ldots, n+2$, $m = 1, 2, \ldots$. Along some subsequence of $m = 1, 2, \ldots$, $\alpha_k^{(m)} \to \alpha_k$ and $\beta_k^{(m)} \to \beta_k$ for $k = 1, \ldots, n+2$, and $\sigma_m = \sigma = \pm 1$. Thus, for $k = 1, 2, \ldots, n+2$, with $I_k = (\alpha_k, \beta_k)$,

$$(-1)^k \sigma (f - \tilde{p}) \geqslant 0 \qquad \text{on} \quad I_k,$$

and

$$(-1)^k \sigma \int_{I_k} (f - \tilde{p})(x)\, dx \geqslant \varliminf_{m \to \infty} |||f_m - p(f_m;\cdot)|||.$$

By (2.5), $\varliminf_{m \to \infty} |||f_m - p(f_m;\cdot)||| \geqslant |||f - \tilde{p}|||$. The polynomial $\tilde{p}(x)$, therefore, satisfies the property of Theorem 3.1. Thus $\tilde{p}(x) \equiv p(f;x)$, and Theorem 4.2 is proved.

We now present analogues of a fundamental result of de La Vallée-Poussin [5].

THEOREM 4.3. *Let* $f \in C[0, 1]$, $n \geqslant 0$, *and* $p \in \pi_n$. *Suppose* $\sigma = \pm 1$, $J_1 < \cdots < J_{n+2}$ *are disjoint (nonempty) open intervals of* $[0, 1]$, *and*

$$(-1)^k \sigma (f - p) \geqslant 0 \qquad \textit{on} \quad J_k, \qquad k = 1, \ldots, n+2.$$

If $p \not\equiv p^*$ *of Theorem* 3.1, *then*

$$|||f - p^*||| > \min_{1 \leqslant k \leqslant n+2} (-1)^k \sigma \int_{J_k} (f - p)(x)\, dx.$$

Remark. It is interesting to note that the last inequality may be false when $p \equiv p^*$, even if we replace $>$ by $\geqslant$. The function f of Fig. 2 (end of Section 3), with $n=0$, $\sigma=-1$, $J_1=(0, 1/2)$, $J_2=(1/2, 1)$, $p \equiv p^* \equiv 0$, is an example.

Proof. Assume the theorem is false. Thus there exist $p \in \pi_n$, $p \not\equiv p^*$, $\sigma = \pm 1$, and $J_1, ..., J_{n+2}$, as above, for which

$$|||f-p^*||| \leqslant (-1)^k \sigma \int_{J_k} (f-p)(x)\,dx, \qquad k=1,..., n+2.$$

We shall prove that, for $k=1,..., n+2$, there exists an $x_k \in J_k$ for which $(-1)^k \sigma(p^*-p)(x_k) \geqslant 0$. As in the beginning of the proof of Theorem 3.1, this leads to a contradiction.

Suppose that there exists a $k \in \{1,..., n+2\}$ for which $(-1)^k \sigma(p^*-p) < 0$ on J_k. Then

$$(-1)^k \sigma(f-p^*)(x) > (-1)^k \sigma(f-p)(x) \geqslant 0$$

for all $x \in J_k$, and therefore

$$|||f-p^*||| \leqslant (-1)^k \sigma \int_{J_k} (f-p)(x)\,dx < (-1)^k \sigma \int_{J_k} (f-p^*)(x)\,dx$$
$$\leqslant |||f-p^*|||.$$

This contradiction proves the theorem.

THEOREM 4.4. *Assume the first two sentences of Theorem* 4.3. *Then*

$$\inf_{q \in \pi_n} |||f-q|||_* \geqslant \min_{1 \leqslant k \leqslant n+2} (-1)^k \sigma \int_{J_k} (f-p)(x)\,dx.$$

Proof. If $p \not\equiv p^*$ of Theorem 3.1, then, from Theorem 4.3 and (2.3),

$$\min_{1 \leqslant k \leqslant n+2} (-1)^k \sigma \int_{J_k} (f-p)(x)\,dx < |||f-p^*|||$$
$$= \inf_{q \in \pi_n} |||f-q|||$$
$$\leqslant \inf_{q \in \pi_n} |||f-q|||_*.$$

If $\inf_{q \in \pi_n} |||f-q|||_*$ is attained, then, by Theorem 3.2, it equals $|||f-p^*|||_*$ which is $\geqslant$ each $(-1)^k \sigma \int_{J_k} (f-p^*)(x)\,dx$.

It therefore remains to consider the case wherein $p \equiv p^*$, but

$\inf_{q \in \pi_n} |||f-q|||_* < |||f-p^*|||_*$. Assume that the theorem does not hold in this case, and choose $\tilde{p} \in \pi_n$ so that

$$|||f-\tilde{p}|||_* < \min_{1 \leqslant k \leqslant n+2} (-1)^k \sigma \int_{J_k} (f-p^*)(x)\, dx \leqslant |||f-p^*|||_*. \quad (4.1)$$

As in the proof of Theorem 4.3, for $k = 1,..., n+2$, there exists an $x_k \in J_k$ for which

$$(-1)^k \sigma(\tilde{p}-p^*)(x_k) \geqslant 0.$$

Thus $\tilde{p} \equiv p^*$ (see the sentence following (3.2)), contradicting (4.1), and the theorem is proved.

We now provide an analogue of Bernstein's comparison theorem [1, p. 8] for $|||\cdot|||$.

THEOREM 4.5. *Let $f, g \in C[0, 1]$, and $n \geqslant 0$. Assume that $f^{(n+1)}$, $g^{(n+1)}$ exist, and $f^{(n+1)} > 0$, throughout* $(0, 1)$. *Assume also that*

$$|g^{(n+1)}(x)| \leqslant f^{(n+1)}(x) \qquad \textit{for all} \quad x \in (0, 1).$$

Then

$$\min_{p \in \pi_n} |||g-p||| \leqslant \min_{p \in \pi_n} |||f-p|||.$$

Proof. Let $p^* \in \pi_n$ satisfy

$$\min_{p \in \pi_n} |||f-p||| = |||f-p^*|||. \quad (4.2)$$

Since $f^{(n+1)} > 0$ on $(0, 1)$, there exist points $0 = x_0 < x_1 < \cdots < x_{n+2} = 1$ such that, for $k = 1,..., n+2$,

$$(-1)^{k+n} (f-p^*) > 0 \qquad \text{on} \quad (x_{k-1}, x_k),$$

and $(-1)^{k+n} \int_{x_{k-1}}^{x_k} (f-p^*)(x)\, dx = |||f-p^*|||$. This follows from Theorem 3.1 and from the fact (an application of Rolle's Theorem) that $f-p^*$ cannot have more than $n+1$ distinct zeros on $[0, 1]$.

Let $\tilde{p} \in \pi_n$ interpolate g at $x_1,..., x_{n+1}$. Then

$$|g-\tilde{p}| \leqslant |f-p^*| \qquad \text{on} \quad [0, 1]. \quad (4.3)$$

Since $(g-\tilde{p})(x_k) = 0$, $k = 1,..., n+1$, (4.3) implies that

$$\min_{p \in \pi_n} |||g-p||| \leqslant |||g-\tilde{p}||| = \max_{1 \leqslant k \leqslant n+2} |||g-\tilde{p}|||_{[x_{k-1}, x_k]}$$

$$\leqslant \max_{1 \leqslant k \leqslant n+2} \int_{x_{k-1}}^{x_k} |(g-\tilde{p})(x)|\, dx$$

$$\leqslant \max_{1 \leqslant k \leqslant n+2} \int_{x_{k-1}}^{x_k} |(f-p^*)(x)|\, dx$$

$$= |||f-p^*||| = \min_{p \in \pi_n} |||f-p|||.$$

(4.3) has nothing whatsoever to do with the gauge $|||\cdot|||$. It is a known result, and may be found, for example, in Kimchi and Richter-Dyn [4].

Remark. It is natural to ask whether Theorem 4.5 remains valid if $|||\cdot|||$ is replaced by $|||\cdot|||_*$ (and min by inf). Since, under the hypotheses of Theorem 4.5 (see its proof), $|||f-p^*||| = |||f-p^*|||_* = \inf\{|||f-p|||_* : p \in \pi_n\}$, while $\min\{|||g-p||| : p \in \pi_n\} \leqslant \inf\{|||g-p|||_* : p \in \pi_n\}$, such a result may be strictly stronger than Theorem 4.5. As a matter of fact, that result is true. However, the above simple method of proof is insufficient to prove it. Our proof, which we omit, is longer, more involved, and similar to that of Theorem 3.2.

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